# Echo Chambers:
# Disagreement in Bayesian Learning

## —Online Appendix—

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This appendix contains the precise statements of many of the theorems referenced in *Echo Chambers: Disagreement in Bayesian Learning.*

The following is a generalization of the Doob Consistency Theorem stated and proved in the elegant text of Ghosh and Ramamoorthi (2003). We retain the notation of the body of the paper.

**Definition 1 (Posterior Consistency).** For each $n$, let $\Pi(\cdot|\mathbf{s}^n)$ be a posterior given $\mathbf{s}^n = (s_1, s_2, ..., s_n)$ and $\Sigma \equiv (S^\infty, \mathcal{B}(S)^\infty)$. The sequence $\{\Pi(\cdot|\mathbf{s}^n)\}$ is consistent at $\theta_0$ if there is $\Sigma_0 \subset \Sigma$ with $F^\infty_{\theta_0}(\Sigma_0) = 1$ such that if $\sigma \in \Sigma_0$ then for every neighborhood of $U$ of $\theta_0$

$$\Pi(U|\mathbf{s}^n(\sigma)) \to 1. \tag{1}$$

**Theorem 2 (Doob's Consistency Theorem).** Suppose that $\Theta$ and $S$ are both complete separable metric spaces endowed with their respective Borel $\sigma$-algebras $\mathcal{B}(\Theta)$ and $\mathcal{B}(S)$ and $\theta \to F_\theta$ be 1-1. Let $\Pi$ be a prior and $\{\Pi(\cdot|\mathbf{s}^n)\}$ be a posterior. Then there exists a $\Theta_0 \subset \Theta$, with $\Pi(\Theta_0) = 1$ such that $\{\Pi(\cdot|\mathbf{s}^n)\}_{n\geq 1}$ is consistent at every $\theta \in \Theta_0$.

The version of the Portmanteau Theorem we employ is stated and proved in `http://www.math.nus.edu.sg/~matsr/ProbI/Lecture5.pdf`.

**Theorem 3 (Portmanteau Theorem).** Let $(\mu_n)_{n\in\mathbb{N}}$ and $\mu$ be probability measures on a metric space $(\Theta, \mathcal{B}(\Theta))$ equipped with the Borel $\sigma$-algebra. The following conditions are equivalent:

1. $\mu_n \Rightarrow \mu$;
2. $\lim_{n\to\infty} \mu_n(A) = \mu(A)$ for all $A \in \mathcal{B}(\Theta)$ with $\mu(\partial A) = 0$, where $\partial A \equiv \bar{A} \cap \bar{A^C}$ denotes the *boundary* of $A$;
3. $\lim_{n\to\infty} \int f \mathrm{d}\mu_n \to \int f \mathrm{d}\mu$ for all bounded real $f$ that are continuous a.e. $\mu$.

As noted in Ghosh and Ramamoorthi (2003), if $\Theta$ is a separable metric space, then it follows from the Portmanteau theorem that consistency of the sequence $\{\Pi(\cdot|\mathbf{s}^n)\}$ at $\theta_0$ is equivalent to requiring that $\Pi(\cdot|\mathbf{s}^n) \Rightarrow \delta_{\theta_0}$ a.e. $F_{\theta_0}$.

The following definition and theorem are drawn from `https://www.ma.utexas.edu/mp_arc/c/02/02-156.pdf`

**Definition 4.** Let $(A, \mathcal{B}(A))$ and $(C, \mathcal{B}(C))$ be Borel spaces. A mapping $f : A \to C$ is called a *Borel mapping* if $f^{-1}(E) \in \mathcal{B}(A)$ for all $E \in \mathcal{B}(C)$. If $f$ is bijective and $f$ and $f^{-1}$ are Borel mappings, then $f$ is called a *Borel isomorphism* .

**Theorem 5 (Borel Isomorphism Theorem).** Suppose $B_1$, $B_2$ are uncountable Borel sets in the *Polish spaces* $A_1$, $A_2$ (separable and completely metrizable). Equip $B_i$ $(i = 1, 2)$ with the relative topology. Then the Borel spaces $(B_1, \mathcal{B}(B_1))$ and $(B, \mathcal{B}(B_2))$ are isomorphic.

**Corollary 6.** Two Polish spaces are Borel-isomorphic if and only if they have the same cardinality.

We thus have a complete list of models for Borel spaces $(A, B(A))$, $A$ Polish:

$(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $(\mathbb{N}, \mathcal{P}(\mathcal{N}))$, and $\big(\{1, 2, ..., n\}, \mathcal{P}(\{1, 2, ..., n\}\big)$, $n \in \mathbb{N}$.

# References

**Ghosh, J.K., and R.V. Ramamoorthi.** 2003. *Bayesian Nonparametrics.* New York:Springer.